Quantum Spin Formulation of the Principal Chiral Model

B. Schlittgen[a] and U.-J. Wiese[b]

[a]Department of Mathematics,
Massachusetts Institute of Technology (MIT), Cambridge, MA 02139, U.S.A.

[b]Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics,
Massachusetts Institute of Technology (MIT), Cambridge, MA 02139, U.S.A.

We formulate the two-dimensional principal chiral model as a quantum spin model, replacing the classical fields by quantum operators acting in a Hilbert space, and introducing an additional, Euclidean time dimension. Using coherent state path integral techniques, we show that in the limit in which a large representation is chosen for the operators, the low energy excitations of the model describe a principal chiral model in three dimensions. By dimensional reduction, the two-dimensional principal chiral model of classical fields is recovered.

1. INTRODUCTION

In [1,2] the construction of QCD as a quantum link model was proposed. In this paper we follow this new approach to quantising field theories, known as *D-theory* to reformulate the principal chiral model. In this approach, classical fields are replaced by quantum operators. In order to compensate for the loss in the number of degrees of freedom brought about by passing from continuous field variables to discrete eigenspectra of the quantum operators, the theory is formulated with an additional dimension, which later disappears; classical fields emerge as low-energy excitations of the discrete variables, provided the $(d+1)$-dimensional theory has massless excitations: When the extent of the extra dimension becomes small in units of the correlation length ξ, the d-dimensional field theory emerges via dimensional reduction. The guiding principles in formulating such a field theory are symmetry considerations. In particular, the quantum operators are Lie group generators chosen so that the Hamiltonian has the same symmetries on a quantum level as the original action has on a classical level.

The virtue of the D-theory formulation of field theories is that the quantum partition function $Z = \mathrm{Tr}\left(\exp\left(-\beta \mathbf{H}\right)\right)$ is a trace over the *discrete* eigenvalues of the quantum operators that make up the Hamiltonian. This promises for a much more efficient treatment by numerical techniques, and it is hoped that cluster algorithms similar to those used for Heisenberg model calculations can be developed.

In the remainder of this article, we formulate the principal chiral model as a quantum spin model. Using coherent state path integral techniques we find an effective action for the low-energy excitations of the model and finally we show how the two-dimensional principal chiral model of classical fields emerges via dimensional reduction.

2. PRINCIPAL CHIRAL MODEL, $d = 2$

2.1. *D-Theory* Formulation

The action of the two-dimensional continuum target theory is the following:

$$S[U] = \frac{1}{2g^2} \int d^2x \mathrm{Tr}\left(\partial_\mu U^\dagger(x) \partial_\mu U(x)\right), \quad (1)$$

where the $U(x)$ are unitary $N \times N$ matrices.

On the lattice, derivatives are replaced by finite differences, and we are free to add and subtract constant terms in the action to obtain a lattice regularised action of the form

$$S_{\mathrm{lat}}[U] = -\frac{1}{g^2} \sum_{<xy>} \mathrm{Tr}\left(U_x^\dagger U_y + V(U_x U_x^\dagger)\right). \quad (2)$$

Note that $U_x U_x^\dagger = I$, so that the potential term is only a constant. Here it has no influence on the physics, but it motivates a similar term in the D-theory Hamiltonian, which will then have some effect on the low energy effective theory of the model.

The target theory has a global $U(N)_L \otimes U(N)_R$ symmetry of the form $U_x \to U_x' = L U_x R^\dagger$, where L and R are unitary matrices. It is known that this symmetry breaks to a $U(N)_V$ symmetry in the ground state.

Let us now replace the classical fields U_x^{ij} by quantum operators $\mathbf{U}_x^{ij}$ and write down a D-theory Hamiltonian, which evolves the two-dimensional system in an additional Euclidean time direction:

$$\mathbf{H} = \frac{J}{2N}\sum_{x,\mu}\left[\mathbf{U}_x^{ij}\left(\mathbf{U}_{x+\mu}^{ij}\right)^\dagger + \mathbf{U}_{x+\mu}^{ij}\left(\mathbf{U}_x^{ij}\right)^\dagger \right.$$
$$\left. + V\left(\mathbf{U}_x^{ij}\left(\mathbf{U}_x^{kj}\right)^\dagger\right) + \text{h.c.}\right]. \qquad (3)$$

We would like this Hamiltonian to have an $U(N)_L \otimes U(N)_R$ symmetry, i.e. $[\mathbf{G}_L^a, \mathbf{H}] = [\mathbf{G}_R^a, \mathbf{H}] = 0$, where $\mathbf{G}_L^a$ and $\mathbf{G}_R^b$ are mutually commuting sets of $U(N)$ generators. This can be realised by embedding $u(N)_L$ and $u(N)_R$ diagonally in the algebra of $U(2N)$ as follows: [2] Let $\{\lambda^a\}$ be matrices of the defining representation of $su(N)$, with commutation relations given by $[\lambda^a, \lambda^b] = 2if_c^{ab}\lambda^c$. Then,

$$\begin{aligned}
\left[\mathbf{G}_L^a, \mathbf{G}_L^b\right] &= 2if_c^{ab}\mathbf{G}_L^c, \quad \left[\mathbf{G}_R^a, \mathbf{G}_R^b\right] = 2if_c^{ab}\mathbf{G}_R^c,\\
\left[\mathbf{G}_R^a, \mathbf{U}^{ij}\right] &= \mathbf{U}^{ik}\lambda_{kj}^a, \quad \left[\mathbf{G}_L^a, \mathbf{U}^{ij}\right] = -\lambda_{ik}^a\mathbf{U}^{kj},\\
\left[\mathbf{G}_R^a, \mathbf{G}_L^b\right] &= [\mathbf{T}, \mathbf{G}_L^a] = [\mathbf{T}, \mathbf{G}_R^a] = 0,\\
\left[\mathbf{T}, \mathbf{U}^{ij}\right] &= 2\mathbf{U}^{ij},\\
\left[\Re\mathbf{U}^{ij}, \Re\mathbf{U}^{k\ell}\right] &= \left[\Im\mathbf{U}^{ij}, \Im\mathbf{U}^{k\ell}\right]\\
&= -i\left(\delta_{ik}\Im\lambda_{j\ell}^a\mathbf{G}_R^a + \delta_{j\ell}\Im\lambda_{ik}^a\mathbf{G}_L^a\right),\\
\left[\Re\mathbf{U}^{ij}, \Im\mathbf{U}^{k\ell}\right] &=\\
i\left(\delta_{ik}\Re\lambda_{j\ell}^a\mathbf{G}_R^a \right. &\left. - \delta_{j\ell}\Re\lambda_{ik}^a\mathbf{G}_L^a + \frac{2}{N}\delta_{ik}\delta_{j\ell}\mathbf{T}\right).
\end{aligned} \qquad (4)$$

If we restrict ourselves to representations of $u(2N)$ which correspond to rectangular Young tableaux with N rows and n_c columns, we can use a fermionic basis of rishons for our representation [2,3]:

$$\begin{aligned}
\mathbf{S}_i^j &= \mathbf{G}_L^{ij} = \sum_x\left(\ell_x^{i\alpha\dagger}\ell_x^{j\alpha} - \frac{n_c}{2}\delta^{ij}\right),\\
\mathbf{S}_{N+i}^{N+j} &= \mathbf{G}_R^{ij} = \sum_x\left(r_x^{i\alpha\dagger}r_x^{j\alpha} - \frac{n_c}{2}\delta^{ij}\right),\\
\mathbf{T} &= \sum_x\left(r_x^{i\alpha\dagger}r_x^{i\alpha} - \ell_x^{i\alpha\dagger}\ell_x^{i\alpha}\right),\\
\mathbf{S}_i^{N+j} &= \mathbf{U}_x^{ij} = \ell_x^{i\alpha}r_x^{j\alpha\dagger},\\
\mathbf{S}_{N+i}^j &= \left(\mathbf{U}_x^\dagger\right)^{ij} = \left(\mathbf{U}_x^{ji}\right)^\dagger = r_x^{i\alpha}\ell_x^{j\alpha\dagger},
\end{aligned} \qquad (5)$$

$$\sum_i\left(\ell^{i\alpha\dagger}\ell^{i\beta} + r^{i\alpha\dagger}r^{i\beta}\right) = \delta^{\alpha\beta}N, \qquad (6)$$

where $\alpha = 1, \ldots, n_c$ is an additional colour index and $i, j = 1, \ldots, N$. For convenience, we have chosen these generators not to be traceless. We then have $\mathbf{G}_{L(R)}^a = \lambda_{ij}^a\mathbf{G}_{L(R)}^{ij}$. The constraint (6) at each lattice point is needed to obtain the correct representation. For $J > 0$, this model is antiferromagnetic and we consider a bipartite lattice, made up of sublattices A and B. For sites on sublattice A we choose one representation, and for sites on sublattice B we choose its conjugate representation. In our case, both representations share the same Young tableau, so they are unitarily equivalent. But they need not be the same representation, as we shall see in the next section. *Note that the properties of* $\mathbf{H}$ *are completely determined, once the representation of* $u(2N)$ *has been specified.* In particular, the physics will be independent of whether the generators are represented by fermionic or bosonic operators.

2.2. Low Energy Effective Theory

Following reference [3] we can set up a coherent state path integral of the form $Z = \int \mathcal{D}Qe^{-S}$, where $S =$

$$\frac{n_c}{4}\int_0^\beta d\tau\int_0^1 du\left[\text{Tr}\left(Q(\tau,u)\frac{\partial Q(\tau,u)}{\partial u}\frac{\partial Q(\tau,u)}{\partial \tau}\right)\right]$$
$$-\int_0^\beta d\tau H(Q(\tau)). \qquad (7)$$

Without going into the details, in this derivation it is important to note that coherent states $|q\rangle$ are labeled by $GL(N,\mathbb{C})$ matrices q and that

$\langle q|\mathbf{S}_\alpha^\beta(x)|q\rangle = \eta_x(n_c/2)Q_\alpha^\beta(x)$. [$\eta_x = +1(-1)$ for x in sublattice $A(B)$.] The Q-field is of the form:

$$Q = \exp\left[\begin{pmatrix} 0 & q \\ -q^\dagger & 0 \end{pmatrix}\right] \times \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} \exp\left[\begin{pmatrix} 0 & -q \\ q^\dagger & 0 \end{pmatrix}\right], \quad (8)$$

where $q \in GL(N,\mathbb{C})$. The following boundary conditions hold:

$$Q(\tau,0) = Q(\tau',0), \quad \text{for all } \tau,\tau'; \\ Q(\tau,1) = Q(\tau); \quad Q(0,u) = Q(\beta,u). \quad (9)$$

We now decompose the matrix field q into a unitary matrix field U and a hermitian matrix field C: $q = CU$. Under $U(N)_L \otimes U(N)_R$ transformations, these fields transform as follows: $q \to q' = LqR^\dagger$, $U \to U' = LUR^\dagger$ and $C \to C' = LCL^\dagger$. Substituting this coset decomposition into (8) we find,

$$Q = \begin{pmatrix} \cos(2C) & -\sin(2C)U \\ -U^\dagger\sin(2C) & -U^\dagger\cos(2C)U \end{pmatrix}. \quad (10)$$

Now define: $B \equiv -\sin(2C)$ and choose $V[(-\sin(2C))UU^\dagger(-\sin(2C))] = V(B^2)$ such that its coefficients are of order one and the minimum of $-3B^2 + V(B^2)$ is attained for B of the form $B = bI,\ 0 < b < 1$. While all other terms in the Hamiltonian are proportional to a^2 (a is the lattice spacing), the term $-3B^2 + V(B^2)$ is not, so that when taking the continuum limit $a \to 0$, any fluctuations of this term around its minimum value are suppressed. The field B is thus frozen in the value bI.

Next, we decompose the field $U(x)$ into staggered and uniform components:

$$U(x) \approx \Omega(x)\sqrt{1 - a^2L^\dagger(x)L(x)} + \eta_x aL(x), \\ \Omega(x)L(x) + L^\dagger(x)\Omega(x) = 0, \quad (11)$$

and $\Omega(x)$ is unitary. Then integrate out the L field to find an effective action for the long-wavelength uniform fluctuations of the form

$$S = \frac{n_c}{2}\eta_x\sum_x\int_0^\beta d\tau \mathrm{Tr}\left(\sqrt{1-b^2}\Omega\partial_\tau\Omega^\dagger\right) \\ + \int_0^\beta d\tau\int d^2x\frac{\rho_s}{2}\mathrm{Tr}\left(\partial_\mu\Omega\partial_\mu\Omega^\dagger + \frac{1}{c^2}\partial_\tau\Omega\partial_\tau\Omega^\dagger\right). \quad (12)$$

Here, $\rho_s = Jb^2n_c^2/2N$ is the spin stiffness and $c = Jb^2n_ca/(N\sqrt{1-b^2})$ is the spin wave velocity. The first term is a Berry Phase term.

We therefore get a low energy effective action for the Goldstone modes associated with the symmetry breaking pattern $U(N)_L \otimes U(N)_R \to U(N)$.

Finally, we assume that the correlation length is much larger than the extent of the additional time dimension, $\xi \gg \beta c$, so that the system is dimensionally reduced to a two-dimensional model. We can then integrate over τ. From [4], we have that for the two-dimensional system, $\xi \propto \exp(2\pi/(g^2N)) = \exp(2\pi\beta\rho_s/N)$, which is indeed consistent with $\xi \gg \beta c$ in the zero-temperature ($\beta \to \infty$) limit.

3. SUMMARY AND CONCLUSIONS

We have reformulated the principal chiral model as a quantum spin model, in which the fields take on discrete values (the eigenvalues of the operators). To compensate for this restriction, we introduced an extra time direction. Having chosen a particular representation for the operators in the Hamiltonian, we were then able to show that the dimensionally reduced theory emerges from the low-energy theory of the collective excitations of the discrete variables. We note that the representation chosen for the operators influences the symmetry breaking pattern.

The extension of these results to quantum link QCD is currently under investigation.

The virtue of the D-theory formulation of field theories is that numerical simulations should be much easier than in conventional field theory formulations, due to the discrete nature of the field variables.

REFERENCES

1. S. Chandrasekharan and U.-J. Wiese, Nucl. Phys. B492 (1997) 455.
2. R. Brower, S. Chandrasekharan and U.-J. Wiese, hep-th/9704106.
3. N. Read and S. Sachdev, Nucl. Phys. B 316 (1989) 609.
4. P.B. Wiegmann, Phys. Lett. 141B (1984) 217.